Modeling Ferrimagnetic Resonators

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Abstract —The impedance matrix for an arbitrary n-port ferrimagnetic resonator is derived by applying Poynting's theorem to a region of space surrounding the resonator. Simplifications to the impedance matrix for low-loss ($Q > \approx 100$) ferrite material make it possible to obtain an equivalent circuit model for the resonator, which can be used with most computer-based circuit simulation programs. The circuit model for the general-case polymodal ferrimagnetic resonator consists of a network of single-pole resonators, each of which has a possible non-frequency-dependent, nonreciprocal phase shift. The components of the circuit model are described in terms of the properties of the ferrite material, and the coupling strength of the microwave circuit to the magnetostatic modes of the ferrimagnet. The method is demonstrated in three simple examples, including a one- and two-port loop coupled filter, and a ferrimagnet in a waveguide.

I. Introduction

FERRIMAGNETIC resonators are microwave circuit components in which the magnetic field of the microwave circuit is magnetically coupled to one or more of the resonant magnetostatic modes of a ferrimagnet which is placed in close proximity to the circuit [1]. These resonators are commonly made with single-crystal YIG and waveguide or microstrip coupling circuits. Ferrimagnetic (YIG) resonators are important components of many microwave devices, especially frequency tunable oscillators and filters. It is important, in the light of recent advances in the computer-aided design (CAD) of microwave circuits, to have an accurate technique of modeling these components. Modeling of specific types of ferrimagnetic resonators has been discussed extensively in the literature [2], but no method of treating the general case has been presented, and in most cases the models are inadequate for accurate simulation with CAD programs. In this paper, an equivalent circuit model is derived which is capable of modeling any type of ferrimagnetic resonator structure, up to high orders of accuracy. The topology of this circuit model is entirely independent of the specific type of microwave circuit structure that is used to couple to these modes, requiring only the specification of $N(2n-1)$ circuit coupling parameters for an n-port circuit coupled to N magnetostatic modes. A straightforward procedure of calculating these parameters up to any order of accuracy is shown. In addition, this technique provides a unique insight into the nature of operation of ferrimagnetic resonators.

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II. Derivation of Impedance Matrix for an n-Port Ferrimagnetic Resonator

To derive the impedance matrix for a general n-port ferrimagnetic resonator, Poynting's theorem is applied to a region of space Ω surrounding the resonator. The volume Ω is chosen so that it entirely encloses the resonator, and its surface S_Ω contains the terminal reference planes of each of the n ports. In addition, Ω must be large enough so that the fields due to the magnetization or polarization of the ferrite[1] material are approximately zero at the surface S_Ω. In the region inside the volume Ω but outside of the volume V which contains the ferrimagnetic material, the fields have the form

$$\vec{H} = H_0 \hat{a}_m + \left(\vec{h}_{\text{app}} + \vec{h}_m\right) e^{j\omega t} \tag{1a}$$

$$\vec{M} = 0 \tag{1b}$$

$$\vec{D} = \epsilon_0 \left(\vec{e}_{\text{app}} + \vec{e}_p\right) e^{j\omega t} \tag{1c}$$

where H_0 is the magnitude; $\hat{a}_m$ is a unit vector in the direction of the dc magnetic field; $\vec{h}_{\text{app}}$ and $\vec{e}_{\text{app}}$ are the applied RF magnetic and electric fields; and $\vec{h}_m$ and $\vec{e}_p$ are the magnetic and electric fields induced by the magnetization and polarization of the ferrimagnetic material. The magnetization and polarization fields are asymptotically dipolar and are assumed to be approximately zero on the surface S_Ω. The magnetization $\vec{M}$ is of course zero in this region.

Inside the volume V which contains the ferrimagnetic material, the fields are given by

$$\vec{H} = H_i \hat{a}_m + \left(\vec{h}_{\text{app}} + \vec{h}_m\right) e^{j\omega t} \tag{2a}$$

$$\vec{M} = M_s \hat{a}_m + (\vec{m}) e^{j\omega t} \tag{2b}$$

$$\vec{D} = \epsilon_0 \epsilon_r \left(\vec{e}_{\text{app}} + \vec{e}_p\right) e^{j\omega t} \tag{2c}$$

where H_i is the internal dc magnetic field, M_s is the dc magnetization (assumed saturated), and $\vec{m}$ is the RF magnetization. The polarization properties of the ferrimagnetic material are assumed linear with a relative dielectric constant of ϵ_r.

Poynting's theorem can now be applied to the volume Ω. Assuming perfect conductors and no radiation loss,

[1]The methods used to derive the impedance matrix here are an extension of those used by Moll. [3]. It is made more general by including the dielectric properties of the ferrite, allowing a more general coupling structure (it is not necessary that conductors be on the boundary of the ferrite) and allowing a different Q (or line width) for each magnetostatic mode.

Poynting's theorem for the RF portion of the time-averaged harmonic fields is

$$\text{average power} + j\cdot\text{reactive power} = \frac{j\omega}{2}\int_{\Omega}(\vec{h}^{*}\cdot\vec{b}-\vec{d}^{*}\cdot\vec{e})\,dV. \quad (3)$$

Expanding the right-hand side, and expressing the power in terms of the currents i_p flowing into the ports $p\in\{1\cdots n\}$ at the surface S_Ω, and the impedance matrix of the resonator Z_{pq},

$$\sum_{p,q=1}^{n}\frac{1}{2}i_p^{*}Z_{pq}i_q=\frac{j\omega}{2}\int_{\Omega}\left(\mu_0|\vec{h}_{\text{app}}|^2-\epsilon_0|\vec{e}_{\text{app}}|^2\right)dV - \frac{j\omega}{2}\int_V\epsilon_0(\epsilon_r-1)|\vec{e}_{\text{app}}|^2\,dV + \frac{j\omega}{2}\int_V\mu_0\left(\vec{h}_{\text{app}}{}^{*}\cdot\vec{m}\right)dV. \quad (4)$$

The terms involving $\vec{h}_{\text{app}}{}^{*}\cdot\vec{h}_m$, $\vec{h}_m{}^{*}\cdot\vec{b}$, and $\vec{e}_{\text{app}}{}^{*}\cdot\vec{e}_p$ make no contribution to the integral, as shown in Appendix I. The first term on the right-hand side of (4) describes the normal reactances of the circuit, as if the ferrite material were not there. The second term describes the additional reactances due to the dielectric properties of the ferrite material (for YIG, $\epsilon_r\approx 16$). The third term describes the ferrimagnetic resonance properties of the circuit. This term is analyzed extensively in Moll [3], and the results are summarized below.

The complete impedance matrix for the general n-port ferrimagnetic resonator can now be easily extracted from (4). The fields $\vec{h}_{\text{app}}$ and $\vec{e}_{\text{app}}$ are related to the currents i_p flowing into the ports:

$$\vec{h}_{\text{app}}=\sum_{p=1}^{n}i_p\vec{k}_p \qquad \vec{e}_{\text{app}}=\sum_{p=1}^{n}i_pZ_{0_p}\vec{l}_p \quad (5)$$

where Z_{0_p} is the characteristic impedance of the transmission line which carries power into port p. The impedance matrix of the resonator is then

$$Z_{pq}=Z_{pq}^{\text{circuit}}+Z_{pq}^{\text{dielectric}}+Z_{pq}^{\text{ferrite}}, \qquad p,q\in\{1\cdots n\} \quad (6a)$$

$$Z_{pq}^{\text{circuit}}=j\omega\int_{\Omega}\left(\mu_0\vec{k}_p^{*}\cdot\vec{k}_q-\epsilon_0Z_{0_p}^{*}Z_{0_q}\vec{l}_p^{*}\cdot\vec{l}_q\right)dV \quad (6b)$$

$$Z_{pq}^{\text{dielectric}}=-j\omega\int_V\left((\epsilon_r-1)\epsilon_0Z_{0_p}^{*}Z_{0_q}\vec{l}_p^{*}\cdot\vec{l}_q\right)dV. \quad (6c)$$

Using the result from Moll [3], the term involving the ferrimagnetic resonance, Z_{pq}^{ferrite}, is a superposition of the impedance matrix for each magnetostatic mode, Z_{pqu}^{ferrite}, $u\in\{1\cdots N\}$:

$$Z_{pq}^{\text{ferrite}}=\sum_{u=1}^{N}Z_{pqu}^{\text{ferrite}}=\sum_{u=1}^{N}j\omega\mu_0V\vec{k}_{pu}^{\dagger}\overleftrightarrow{\chi}_u\vec{k}_{qu}. \quad (6d)$$

The *coupling vectors* $\vec{k}_{pu}$ are defined as

$$\vec{k}_{pu}=\begin{pmatrix}k_{pur}\\ k_{pui}\end{pmatrix}=\frac{1}{V}\int_V\begin{pmatrix}\vec{k}_p\cdot\vec{\psi}_{ur}\\ \vec{k}_p\cdot\vec{\psi}_{ui}\end{pmatrix}dV, \qquad \vec{\psi}_{ur}=\frac{\vec{\psi}_u+\vec{\psi}_{-u}}{\sqrt{2}},$$
$$\vec{\psi}_{ui}=\frac{\vec{\psi}_u-\vec{\psi}_{-u}}{\sqrt{2}}, \qquad \vec{\psi}_u^{*}=\vec{\psi}_{-u} \quad (6e)$$

where $\vec{\psi}_u$ is the uth magnetostatic mode eigenvector [4]. The coupling vectors $\vec{k}_{pu}$ are seen to be proportional to the projection of the magnetic field of the circuit structure onto the magnetostatic modes of the ferrite. $\overleftrightarrow{\chi}_u$ is the susceptibility tensor of the uth magnetostatic mode of the ferrimagnet:

$$\overleftrightarrow{\chi}_u=\frac{1}{\omega_u^2-\left(\omega-\frac{1}{2}j\Delta\omega_u\right)^2}\cdot\begin{pmatrix}\omega_u\omega_M & -j\left(\omega-\frac{1}{2}j\Delta\omega_u\right)\omega_M\\ j\left(\omega-\frac{1}{2}j\Delta\omega_u\right)\omega_M & \omega_u\omega_M\end{pmatrix} \quad (6f)$$

where $\omega_M=\gamma M_s$, and γ is the gyromagnetic ratio. As Moll describes, this susceptibility tensor is that of a general shape ferrimagnet with the magnetostatic modes, $\vec{\psi}_{ur}$, $\vec{\psi}_{ui}$, taken as the basis vectors. Note that the phenomenological damping term that Moll used, $\Delta\omega$, has been replaced with $\Delta\omega_u$, as justified in Appendix II. This allows each magnetostatic mode to have unique loss characteristics.

Equations (6) describe the complete impedance matrix for any n-port ferrimagnetic resonator. In the next section, these equations are even further simplified by assuming that the ferrite material is relatively low-loss. This allows derivation of a simple equivalent circuit model of the ferrimagnetic resonator that is valid in the general case.

III. The Impedance Matrix for Low-Loss Ferrimagnetic Resonators

The impedance matrix of each magnetostatic mode of the resonator, Z_{pqu}^{ferrite}, can be greatly simplified by making the assumption that the ferrimagnetic material is relatively low-loss, so that $Q_u>\approx 100$ for each magnetostatic mode $u\in\{1\cdots N\}$ to be considered.[2] This approximation is not a very restrictive assumption for most applications.

Introducing the quality factor Q_u of each mode u, the susceptibility tensor $\overleftrightarrow{\chi}_u$ can be rewritten in terms of Q_u. Then keeping only terms that are first order in $(1/Q_u)$, the susceptibility tensor becomes

$$Q_u\equiv\frac{\omega_u}{\Delta\omega_u} \quad (7)$$

$$\overleftrightarrow{\chi}_u=\frac{1}{\omega_u^2-\omega^2+j\dfrac{\omega\omega_u}{Q_u}}\begin{pmatrix}\omega_u\omega_M & -j\omega\omega_M\\ j\omega\omega_M & \omega_u\omega_M\end{pmatrix}. \quad (8)$$

[2] For reference, $Q_1>\approx 1000$ for the uniform magnetostatic mode of an ellipsoidal single-crystal YIG resonator.

The magnitude of $\overleftrightarrow{\chi}_u$ and, therefore, the magnitude of Z_{pqu}^{ferrite} are negligible for frequencies that are not very close to ω_u. Letting $\omega = \omega_u + \delta_u \omega_u$, and keeping only terms first order in δ_u and $1/Q_u$,

$$\delta_u \equiv \frac{\omega - \omega_u}{\omega_u} \tag{9}$$

$$\overleftrightarrow{\chi}_u = \frac{-j(\omega_M/\omega_u)Q_u}{(1 + j2Q_u\delta_u)} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}. \tag{10}$$

Using the low-loss approximation of the susceptibility tensor (10) in the definition of Z_{pqu}^{ferrite} (eq. (6d)), the following simplified result is obtained:

$$Z_{pqu}^{\text{ferrite}} = \frac{\omega_M V Q_u}{(1 + j2Q_u\delta_u)} \mu_0 \vec{k}_{pu}^{\dagger} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix} \vec{k}_{qu}. \tag{11}$$

Note that the term on the right-hand side of (11), involving the coupling vectors only, can now be separated from the rest of the equation. It becomes very useful to introduce some new quantities, the *coupling tensor* K_{pqu}, the *coupling magnitude* $|K_{pqu}|$, and the *coupling phase shift* Φ_{pqu}, which can be defined from this term. Typically, the vectors $\vec{k}_{pu}$ are functions of frequency, but any frequency dependence would be second order in δ_u, so each component u of the tensor is evaluated at the resonant frequency ω_u:

$$K_{pqu} \equiv |K_{pqu}| e^{-j\Phi_{pqu}} \equiv \mu_0 \vec{k}_{pu}^{\dagger} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix} \vec{k}_{qu}\big|_{\omega = \omega_u}. \tag{12}$$

Note that the coupling tensor K_{pqu} is dependent only on the geometry of the circuit structure, and of the ferrimagnet, while the rest of the terms in (11) involve no circuit-dependent factors. Therefore, a separation of the geometrical considerations of circuit coupling from the general ferrite material properties has been achieved. The coupling tensor has some properties which prove to be very important in determining the final circuit model of the resonator. Their proof is in Appendix III, and the properties are exhibited here:

$$K_{pqu} = \frac{K_{ptu} K_{tqu}}{K_{ttu}} \qquad K_{pqu}^{*} = K_{qpu},$$
$$p, q, t \in \{1 \cdots n\}, \; u \in \{1 \cdots N\} \tag{13a}$$

$$|K_{pqu}| = |K_{qpu}| = \sqrt{|K_{ppu}||K_{qqu}|},$$
$$p, q \in \{1 \cdots n\}, \; u \in \{1 \cdots N\} \tag{13b}$$

$$\Phi_{ppu} = 0 \qquad \Phi_{pqu} = -\Phi_{qpu} \qquad \Phi_{pqu} = \Phi_{ptu} + \Phi_{tqu},$$
$$p, q, t \in \{1 \cdots n\}, \; u \in \{1 \cdots N\}. \tag{13c}$$

It is also useful to define a ferrimagnetic material dependent term, the *ferrimagnetic resistance* of the uth magnetostatic mode, which is a pure real number proportional to the product of the total magnetization of the ferrimagnet and the Q of that mode, and has units of impedance:

$$\rho_u \equiv \omega_M V Q_u = \gamma M_s V Q_u. \tag{14}$$

Using the definitions (12) and (14) in (11), a very simple expression for the impedance matrix of the uth magnetostatic mode can be obtained:

$$Z_{pqu}^{\text{ferrite}} = \frac{\rho_u |K_{pqu}|}{(1 + j2Q_u\delta_u)} e^{-j\Phi_{pqu}}. \tag{15}$$

This is seen to be in the form of the impedance matrix for an inductively coupled single-pole resonator with an additional non-frequency-dependent phase shift. The coupling to the resonator is proportional to the product of the ferrimagnetic resistance and the coupling magnitude, while the non-frequency-dependent phase shift is given by the coupling phase shift. The resonant frequency and the Q of the resonator are those of the uth magnetostatic mode. A circuit model for this impedance matrix will be derived in the next section.

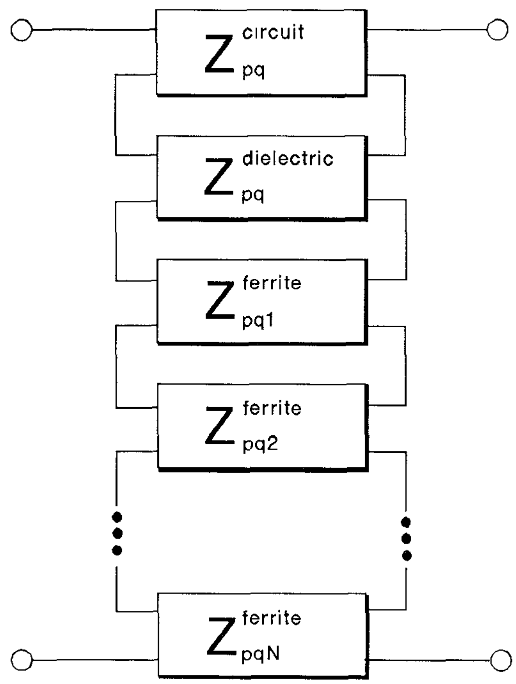

Fig. 1. The 2-port ferrimagnetic resonator is a collection of series-connected 2-ports. One 2-port is that of the circuit without the ferrimagnet, the next represents the additional reactances due to the dielectric properties of the ferrimagnet, and there is one 2-port for each of the N magnetostatic modes.

IV. Equivalent Circuit Model of an n-Port Ferrimagnetic Resonator

The total impedance matrix (eq. (6a)) for the n-port resonator is a summation of matrices, each of which is itself an impedance matrix for an n-port element. The summation of the matrices implies that each of these n-port elements is connected in series to obtain the total n-port circuit for the resonator. This situation is illustrated in Fig. 1 for a 2-port resonator. To model the complete resonator is equivalent to modeling each of the elements Z_{pq}^{circuit}, $Z_{pq}^{\text{dielectric}}$, and Z_{pqu}^{ferrite}, $u \in \{1 \cdots N\}$.

The first term in (6a), Z_{pq}^{circuit}, just represents the normal circuit impedance with the ferrite absent. We assume that a circuit model can be obtained for the circuit impedance. The second term, $Z_{pq}^{\text{dielectric}}$, represents additional reactances due to the dielectric loading of the circuit by the ferrimagnetic material. The dielectric loading is usually ignored by most models, but it can be significant if the size of the ferrimagnet is comparable to the size of the coupling circuit geometry.

The simple form of (15) and the symmetry of the relations (13) suggest that it should be possible to derive a simple circuit model which will mirror the impedance matrix Z_{pqu}^{ferrite} for each magnetostatic mode $u \in \{1 \cdots N\}$, and indeed this is the case. By first demonstrating the circuit representation for the 2-port, it is easy to deduce the form of the general n-port model.

The two port impedance matrix $\boldsymbol{Z}_u^{\text{ferrite}}$ for the mode u is

$$\boldsymbol{Z}_u^{\text{ferrite}} = \frac{\rho_u}{(1+j2Q_u\delta_u)} \cdot \begin{pmatrix} |K_{11u}| & \sqrt{|K_{11u}||K_{22u}|}\, e^{-j\Phi_{12u}} \\ \sqrt{|K_{11u}||K_{22u}|}\, e^{j\Phi_{12u}} & |K_{22u}| \end{pmatrix} \quad (16)$$

By transforming this into the transmission matrix representation, it is easy to separate this into cascaded elements, one of which represents the phase shift $e^{-j\Phi_{pqu}}$ and the other the resonance. After transforming and factoring, the transmission matrix is

$$\boldsymbol{T}_u^{\text{ferrite}} = \begin{pmatrix} e^{-j\Phi_{12u}} & 0 \\ 0 & e^{-j\Phi_{12u}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\dfrac{|K_{11u}|}{|K_{22u}|}} & 0 \\ \dfrac{(1+j2Q_u\delta_u)/\rho_u}{\sqrt{|K_{11u}||K_{22u}|}} & \sqrt{\dfrac{|K_{22u}|}{|K_{11u}|}} \end{pmatrix}. \quad (17)$$

It is now straightforward to model the two cascaded elements. The first element is a nonreciprocal phase shift, which, upon transforming into H matrix form, yields the following relations between the currents and voltages at ports 1 and 2 of the element:

$$\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & e^{-j\Phi_{12u}} \\ -e^{j\Phi_{12u}} & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ V_2 \end{pmatrix}. \quad (18)$$

This nonreciprocal phase shift element can be modeled with a voltage source and a current source, as shown in Fig. 2(a). The second element has an impedance matrix of the form of (16) with $\Phi_{pqu} = 0$, which is the impedance matrix of an inductively coupled single-pole resonator.[3] The resonant frequency of the resonator is equal to ω_u, and the Q of the resonator is equal to Q_u. The relation between the resistor in the resonator and the mutual inductance is indicated in Fig. 2(b). The complete circuit model of the ferrimagnetic resonance element of the 2-port ferrimagnetic resonator is shown in Fig. 2(b).

[3]Technically, the outer inductor of each coupled inductor pair must have zero inductance for this impedance matrix to be correct. But usually, an inductance in Z_{pq}^{circuit} can be combined with the outer inductor to yield a total inductance which is nonzero. See examples 1 and 2. If this is not the case, current sources can be used to model the coupled inductors.

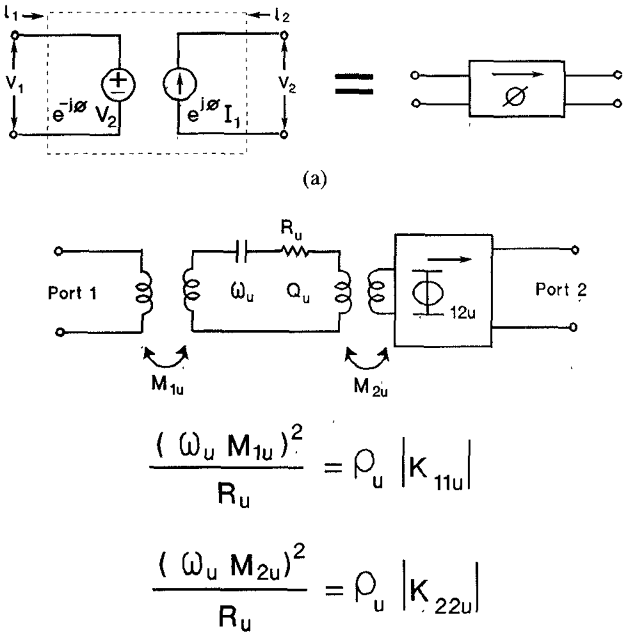

Fig. 2. (a) The nonreciprocal phase shift element consists of a voltage and a current source with the indicated transfer functions. (b) The complete 2-port circuit model for Z_{pqu}^{ferrite} is the phase shift element cascaded with an inductively coupled resonator.

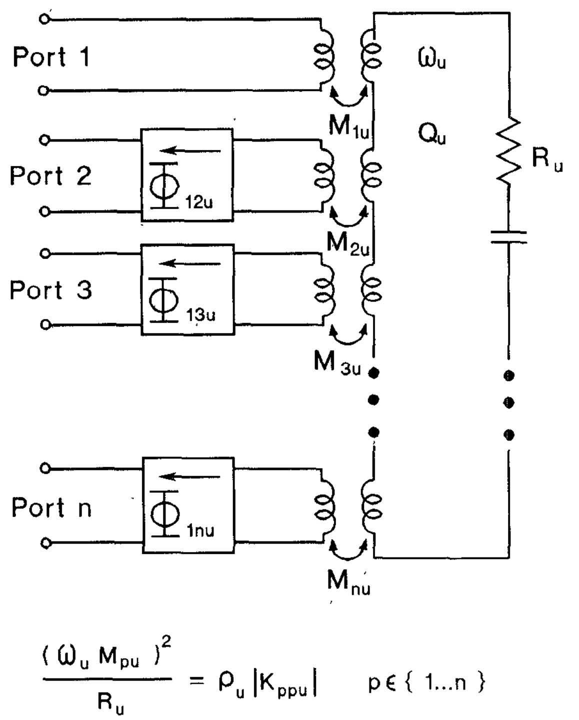

Fig. 3. The complete n-port circuit model for the uth magnetostatic mode resonance, Z_{pqu}^{ferrite}.

The general n-port ferrimagnetic resonance element, Z_{pqu}^{ferrite}, is shown in Fig. 3. It is easily deduced from the similarity to the 2-port case, and by using the relations (13c) for the nonreciprocal phase shift, that

$$\Phi_{pqu} = \Phi_{1qu} - \Phi_{1pu}, \qquad p, q \in \{1 \cdots n\} \quad u \in \{1 \cdots N\}. \quad (19)$$

The circuit in Fig. 3, when connected in series with Z_{pq}^{circuit} and $Z_{pq}^{\text{dielectric}}$, can then model the ferrimagnetic resonance of any n-port circuit structure when the parameters indicated in the figure are supplied. The quantities necessary to calculate the circuit element values fall into two categories:

$$\text{ferrite material properties: } \omega_u, \quad Q_u, \quad \rho_u = \gamma M_s V Q_u \qquad u \in \{1 \cdots N\} \tag{20a}$$

$$\text{circuit coupling characteristics: } |K_{ppu}|, \quad \Phi_{1pu} \qquad p \in \{1 \cdots n\}, u \in \{1 \cdots N\}. \tag{20b}$$

Typically, the ferrite material properties, Q_u and M_s, would be measured on a calibrated test station for which the circuit coupling characteristics are known. The resonant frequency of each mode, ω_u, is usually a known function of the dc biasing field and the saturation magnetization M_s [4]. The circuit coupling characteristics can be calculated up to any degree of accuracy with (12) and (6e) if the spatial dependence of the RF magnetic field and the magnetostatic mode is known. In most cases, evaluation of (6e) may require numerical integration; however a simpler case can usually be considered to obtain the semiquantitative circuit coupling characteristics (see the examples below).

Figs. 4 and 5 illustrate the complete circuit model for the 1- and 2-port cases, when coupled to N magnetostatic modes. In the next section this method of ferrimagnetic resonator modeling is applied to specific cases.

V. Examples

Example 1: Loop Coupled 1-Port Ferrimagnetic Resonator

Consider a ferrimagnet of small volume V in the center of a circular wire loop of radius r (see Fig. 6(a)). The dc magnetic biasing field is uniform and is in the $\hat{z}$ direction, and the axis of the loop is in the $\hat{x}$ direction. Only coupling to the uniform magnetostatic mode $\vec{\psi}_1$ will be considered. The uniform magnetostatic mode eigenvector is simply

$$\vec{\psi}_1 = \frac{1}{\sqrt{2}}(\hat{x} + j\hat{y}) \qquad \vec{\psi}_{1r} = \hat{x} \qquad \vec{\psi}_{1i} = \hat{y}. \tag{21}$$

The RF magnetic field $\vec{h}_1$ at the center of the loop, when a current i_1 is flowing into the port, is

$$\vec{k}_1 = \frac{\vec{h}_1}{i_1} = \frac{1}{2r}\hat{x}. \tag{22}$$

Assuming that $V \ll r^3$, the integration in (6e) is readily performed to calculate the coupling vector,

$$\vec{k}_{11} = \begin{pmatrix} k_{11r} \\ k_{11i} \end{pmatrix} = \begin{pmatrix} 1/2r \\ 0 \end{pmatrix}. \tag{23}$$

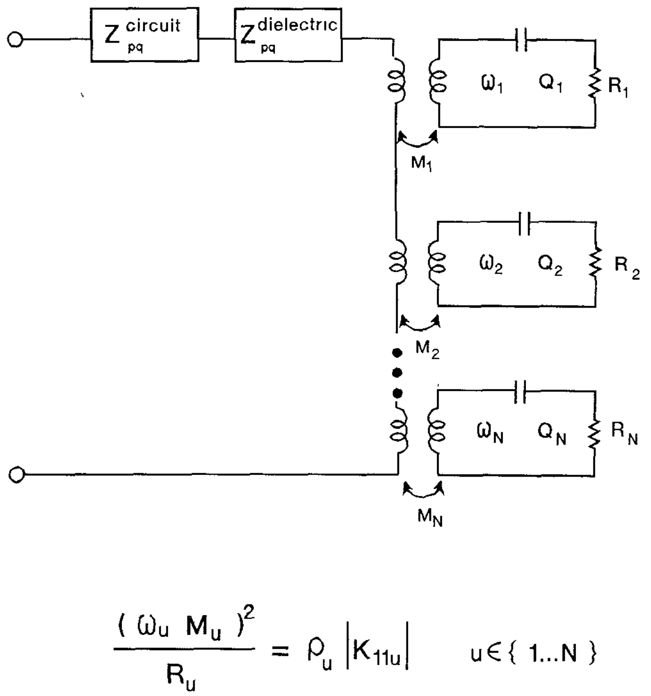

Fig. 4. The complete circuit model for a 1-port ferrimagnetic resonator which is coupled to N magnetostatic modes.

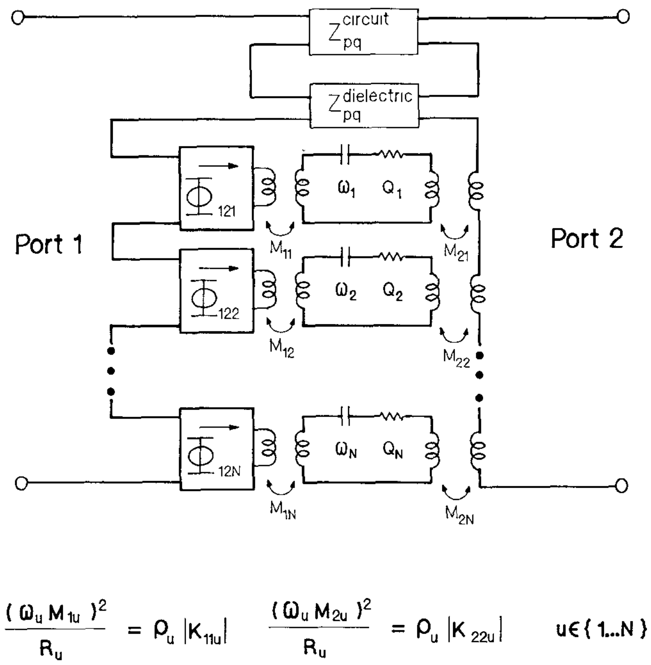

Fig. 5. The complete circuit model for a 2-port ferrimagnetic resonator which is coupled to N magnetostatic modes.

And now, the coupling tensor, K_{111} (which is just a scalar in this case since $n = N = 1$), is given by (12):

$$K_{111} = |K_{111}| = \frac{\mu_0}{4r^2}. \tag{24}$$

The equivalent circuit is shown in Fig. 6(b). The loop is modeled by an inductor L_{11}, and the dielectric properties are ignored, since $V \ll r^3$. This model agrees with Carter's result [5].

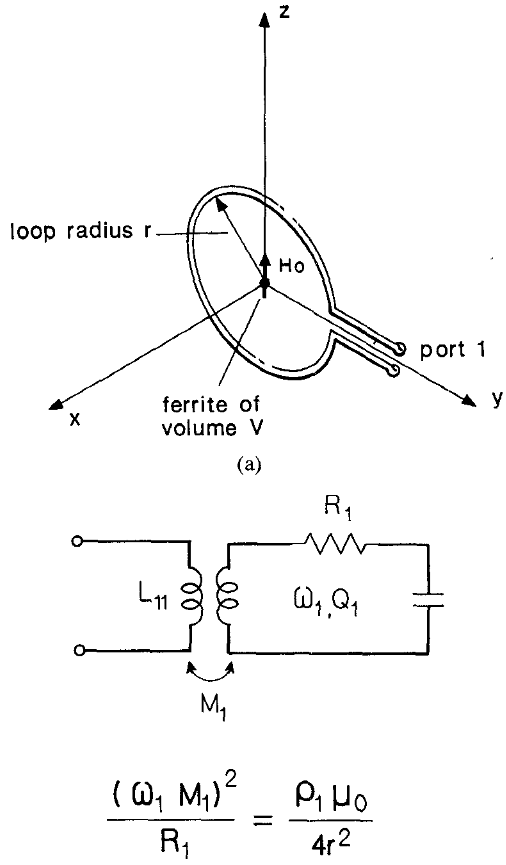

Fig. 6. (a) The geometry for example 1, a 1-port loop coupled ferrimagnetic resonator. (b) The equivalent circuit.

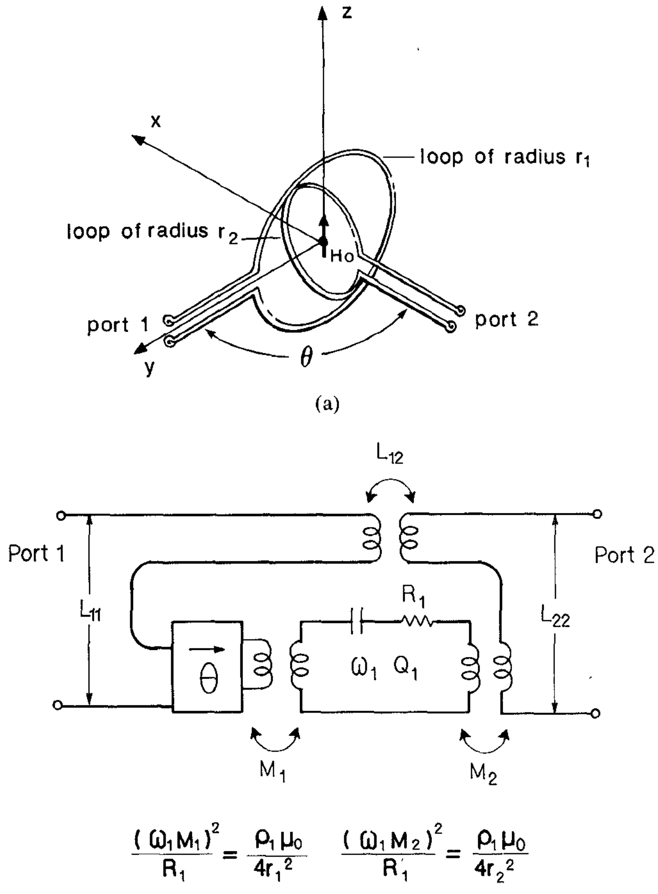

Fig. 7. (a) The geometry for example 2, a 2-port loop coupled ferrimagnetic resonator. (b) The equivalent circuit.

Example 2: 2-Port Loop Coupled Ferrimagnetic Resonator/Filter

Consider a ferrimagnet of small volume V at the center of two circular wire loops of radii r_1 and r_2 (see Fig. 7(a)). The biasing field is in the $\hat{z}$ direction, and the axis of loop 1 is in the $\hat{x}$ direction. The axis of loop 2 is at an angle θ towards $\hat{y}$ with respect to the axis of loop 1. Again, only coupling to the uniform magnetostatic mode is considered. The uniform magnetostatic mode eigenvector is the same as in example 1. The magnetic fields at the center of the loop are

$$\vec{k}_1 = \frac{\vec{h}_1}{i_1} = \frac{1}{2r_1}\hat{x} \qquad \vec{k}_2 = \frac{\vec{h}_2}{i_2} = \frac{1}{2r_2}(\cos(\theta)\hat{x} + \sin(\theta)\hat{y}). \tag{25}$$

Again, the integration in (6e) is trivial, since $V \ll r^3$, so the coupling vectors are

$$\vec{k}_{11} = \frac{1}{2r_1}\begin{pmatrix}1\\0\end{pmatrix} \qquad \vec{k}_{21} = \frac{1}{2r_2}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}. \tag{26}$$

The coupling tensor is calculated with (12):

$$|K_{111}| = \frac{\mu_0}{4r_1^2} \qquad |K_{221}| = \frac{\mu_0}{4r_2^2} \qquad \Phi_{121} = \theta. \tag{27}$$

It is interesting to note that the nonreciprocal phase shift angle Φ_{121} is the same as the physical angle θ. The equivalent circuit is shown in Fig. 7(b), where the loops are modeled by coupled inductors. For the case of orthogonal loops ($\theta = \pi/2$), the result reduces to that given by Carter [5].

Example 3: Ferrimagnet in Rectangular Waveguide Propagating TE_{10} Mode

Consider a ferrimagnet of small volume V inside a length l of rectangular waveguide of width a and height b. The ferrite is located at a distance x_0 from the sidewall of height b, centered in the y direction and at an arbitrary z location (see Fig. 8(a)). The assumption is made that only the TE_{10} waveguide mode is propagating and that only the uniform magnetostatic mode is excited. The RF magnetic fields, $\vec{h}_1$ and $\vec{h}_2$ at the ferrimagnet location, are then given by

$$\vec{k}_1 = \frac{\vec{h}_1}{i_1} = \sqrt{\frac{2}{ab}}\,(\hat{x} - j\epsilon(\omega, x)\hat{y})\sin(\kappa x)e^{j\beta l/2}$$

$$\vec{k}_2 = \frac{\vec{h}_2}{i_2} = -\sqrt{\frac{2}{ab}}\,(\hat{x} + j\epsilon(\omega, x)\hat{y})\sin(\kappa x)e^{j\beta l/2} \tag{28}$$

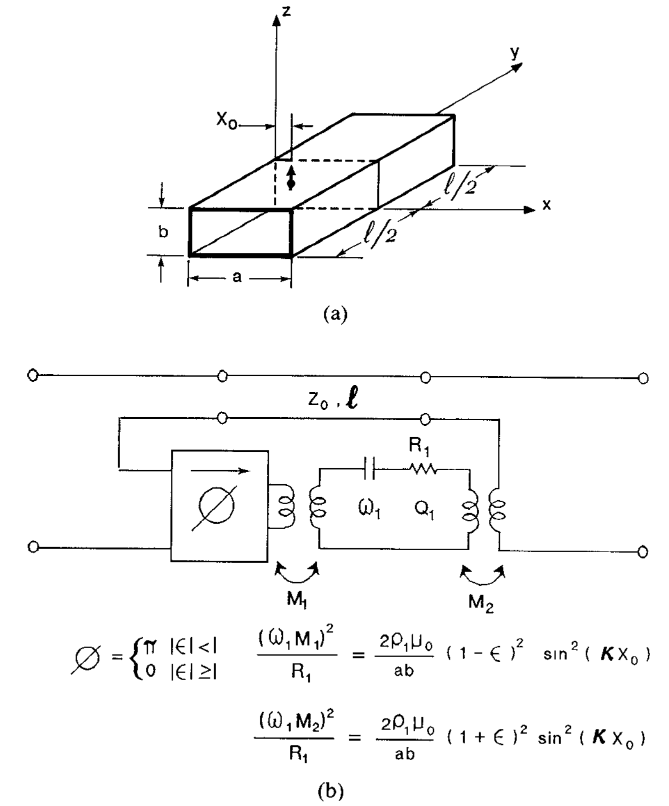

Fig. 8. (a) The geometry for example 3, a ferrimagnet in a waveguide. (b) The equivalent circuit.

where

$$\epsilon(\omega, x) \equiv \frac{\kappa}{\beta}\cot(\kappa x) \qquad \kappa \equiv \frac{\pi}{a} \qquad \beta^2 \equiv \left(\frac{\omega}{c}\right)^2 - \kappa^2.$$

The coupling vectors are again obtained from (6e), after a trivial integration (we again assume that $V \ll a^3$):

$$\vec{k}_{11} = \sqrt{\frac{2}{ab}}\sin(\kappa x_0)\begin{pmatrix} 1 \\ -j\epsilon(\omega, x_0)\end{pmatrix}$$

$$\vec{k}_{21} = -\sqrt{\frac{2}{ab}}\sin(\kappa x_0)\begin{pmatrix} 1 \\ j\epsilon(\omega, x_0)\end{pmatrix}. \tag{29}$$

And the coupling tensor is calculated from these with (12):

$$K_{111} = \frac{2\mu_0}{ab}(1-\epsilon(\omega_1, x_0))^2\sin^2(\kappa x_0)$$

$$K_{221} = \frac{2\mu_0}{ab}(1+\epsilon(\omega_1, x_0))^2\sin^2(\kappa x_0)$$

$$K_{121} = \frac{2\mu_0}{ab}\left(\epsilon(\omega_1, x_0)^2 - 1\right)\sin^2(\kappa x_0)$$

$$\Phi_{121} = \begin{cases} 0, & |\epsilon(\omega_1, x_0)| \geq 1 \\ \pi, & |\epsilon(\omega_1, x_0)| < 1. \end{cases} \tag{30}$$

The waveguide can be modeled as a transmission line of impedance Z_0 and length l, an element which is available on most circuit simulation programs. The complete equivalent circuit model is shown in Fig. 8(b). The result agrees with that of James [6].

VI. Summary

A lumped element circuit model has been derived for a general n-port ferrimagnetic resonator which is coupled to N magnetostatic modes. The circuit model requires the specification of $N(2n-1)$ coupling parameters, $|K_{ppu}|$, Φ_{1pu}, and $3N$ material parameters ω_u, ρ_u, Q_u. The material parameters must be measured or estimated, and methods for calculating the coupling parameters up to any order of accuracy have been shown. The variety and complexity of the many types of ferrimagnetic resonators have been reduced to the specification of these parameters in the circuit model. The technique is straightforward, and is capable of modeling any ferrimagnetic resonator. The only approximations that have been made are that the material is low-loss ($Q_u > \approx 100$, $u \in \{1 \cdots N\}$) and that only N magnetostatic modes are excited. Therefore this method is appropriate for modeling common single-crystal YIG resonators.

The use of the technique has been demonstrated in some simple examples, and application to more accurate models of more complex resonators should be evident. This technique should prove very valuable in the development of more accurate computer-aided design models of ferrimagnetic resonators.

Appendix I

To demonstrate that the integrals in Poynting's theorem with the terms $\vec{h}^*_{\text{app}} \cdot \vec{h}_m$, $\vec{h}^*_m \cdot \vec{b}$, and $\vec{e}^*_{\text{app}} \cdot \vec{e}_p$ are zero a proof similar to Moll's [3] is used, except it is not required that the surface S_Ω be a conductor boundary. To begin with, note that $\nabla \cdot \vec{b} = \nabla \cdot \vec{h}_{\text{app}} = \nabla \cdot \vec{e}_{\text{app}} = 0$ and that $\vec{h}_m = \nabla \phi_m$ and $\vec{e}_p = \nabla \phi_p$, since the static approximation is assumed for these asymptotically dipolar fields, $V \ll (\omega/c)^3$. Then the volume integrals can be transformed into surface integrals on S_Ω, where we have assumed that $\vec{h}_m = 0$, $\vec{e}_p = 0$. Then, on S_Ω, $\phi_m = \phi_p = constant$, which we choose to be zero. For example,

$$\int_\Omega \vec{h}^*_m \cdot \vec{b}\, dV = \int_{S_\Omega} \phi^*_m(\vec{b} \cdot \hat{n})\, dA - \int_\Omega \phi^*_m(\nabla \cdot \vec{b})\, dV = 0. \tag{31}$$

Appendix II

It can be easily seen experimentally that the different magnetostatic modes can have very different Q values. This can be taken into account in the theory, by introducing a phenomenological damping operator $\boldsymbol{A}_{\text{op}}$ which has the magnetostatic modes as eigenvectors, and the half-width of the modes as eigenvalues. The representation of this operator is not important as long as the eigenvectors and eigenvalues are known:

$$\boldsymbol{A}_{\text{op}} \vec{\psi}_{\pm u} = \frac{1}{2}\Delta\omega_u \vec{\psi}_{\pm u}. \tag{32}$$

Using this in place of the scalar phenomenological damping term in the small-signal gyromagnetic equation, from which Moll derives his results, yields the following equation:

$$(\boldsymbol{L}_{\mathrm{op}} - \boldsymbol{A}_{\mathrm{op}})\vec{m} = j\omega\vec{m} - \gamma M_s \hat{a}_m \times \vec{h}_{\mathrm{app}} \tag{33}$$

where $\boldsymbol{L}_{\mathrm{op}}$ is the operator described by Moll. The rest of Moll's derivation is the same, and leads directly to the modified susceptibility tensor, equation (6f).

APPENDIX III

Only the first relation in the set of equations (13) is proved. The rest follow easily. To start, note the following relation:

$$\left[\vec{k}_{pu}, \vec{k}^{\dagger}_{pu}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}\right] = \left(k^{*}_{pur} + jk^{*}_{pui}\right)\begin{pmatrix} j \\ 1 \end{pmatrix}\begin{pmatrix} k_{pui} & -k_{pur} \end{pmatrix} \tag{34}$$

where the brackets [,] indicate the commutator. Using this in the following relation:

$$\begin{aligned} K_{ptu}K_{tqu} &= \vec{k}^{\dagger}_{pu}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}\vec{k}_{tu}\vec{k}^{\dagger}_{tu}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}\vec{k}_{qu} \\ &= \vec{k}^{\dagger}_{pu}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}\vec{k}^{\dagger}_{tu}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}\vec{k}_{tu}\vec{k}_{qu} + \vec{k}^{\dagger}_{pu} \\ &\quad \times \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}\left[\vec{k}_{pu}, \vec{k}^{\dagger}_{pu}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}\right]\vec{k}_{qu} \\ &= K_{pqu}K_{ttu}, \qquad p,q,t \in \{1 \cdots n\} \end{aligned} \tag{35}$$

since the second term in the summation is zero.

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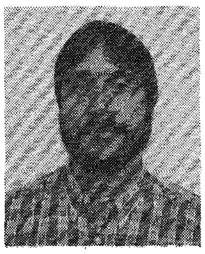

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